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Optimal Policy for Deteriorating Items with Power Demand and Shortages

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Abstract: This paper analyzes the inventory replenishment policy of an economic ordering quantity model for deteriorating items whose demand follows a power demand pattern. The inventory system studied here considers that the deterioration rate of the product is a three-parameter Weibull distribution and the unsatisfied demand is partially backlogged at a constant rate. Thus, the model is developed under the assumption of that at the beginning of the inventory cycle there are no shortages and the deterioration process starts after a known and fixed time (life-time of the product). During that period, the inventory level depletes due to demand only. After that life-item period, deterioration takes place. The inventory level gradually decreases due to demand and deterioration of items and falls to zero level. Thereafter, shortages occur and are partially backlogged. The backlogged demand is satisfied at the beginning of the new inventory cycle when a new replenishment arrives to the inventory. The objective is to minimize the total cost per unit time of the inventory system. This cost includes the ordering cost, the holding cost, the deteriorating cost, the backlogging cost and the lost sales cost. An algorithmic procedure is proposed to find the optimal policy of the described model. Some numerical examples are provided to illustrate the developed model and the solution procedure.

Key words: *Inventory Systems, Power demand Pattern, Partial backlogging, Weibull Deterioration.*

INTRODUCTION

Inventory Theory is one of the most interesting fields of Management Science and Operations Research. This field includes a wide number of mathematical models and techniques that allow determining the best decisions related to the management, replenishment and maintenance of inventory systems. The first mathematical model of inventory control is known as the economic order quantity (EOQ) model and was proposed by Harris [1] at the beginning of the twentieth century. Regardless of its simplicity, that model is still applied in industry and commerce nowadays. A lot of research has since been carried out to enhance that model by modifying its unrealistic assumptions. For example, in the classical EOQ model the demand rate is assumed as a constant. However, in practice, customer´s demand is changing with time. Thus, inventory models with time-varying demand are need to represent appropriately the evolution of the inventory level. Many researchers have analyzed inventory models where the demand rate varies with time. Thus, Ritchie [2] developed a solution procedure for an inventory model with linear increasing demand. Yang et al. [3] studied an approach to analyze an inventory system with non-linear decreasing demand. Sakaguchi [4] developed an inventory policy for a system with time-varying demand.

When products are held in stock, the deterioration of items may take place in the inventory system. Deterioration is considered as damage or decay in the quality of products kept in inventory. Different inventory models for deteriorating items has been

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studied by several authors in the last years. Ghare and Schrader [5] developed one of the first papers on deterioration inventory models, assuming an exponentially decaying inventory. Misra [6] studied an economic production quantity model for deteriorating items. Dave and Patel [7] analyzed an inventory model with constant deterioration rate and a linear demand rate. Raafat [8], Goyal and Giri [9] and Li et al. [10] proposed several surveys on inventory models for deteriorating items. Some researchers have considered that items have a life-time in which products maintain its quality. Thus, the deterioration of items kept in inventory stars after that period. In this line, there are the papers of Wu et al. [11], Triarthy and Pradhan [12] and Gupta and Arora [13].

If shortages occur during the inventory cycle, then some customers make decision no wait and they buy the items from other sellers, while other are willing to wait for backorder. Several works on inventory models have assumed partial backlogging. For example, San José et al. [14], Chang and Dye [15], and Mishra and Singh [16] studied several inventory models considering partial backlogging. Recently, San José et al. [17] studied the optimal policy for an inventory system with a power demand pattern assuming partial backlogging.

In the literature of inventory control some researchers have studied inventory systems taking into account different time-dependent demands and deteriorations. Goel and Aggarwal [18] presented an inventory model for deteriorating items with a power demand pattern, assuming a constant deterioration rate. Datta and Pal [19] developed an inventory system with a power demand for items with a variable deterioration rate. Lee and Wu [20] investigated an EOQ model for deteriorating items considering a Weibull deterioration distribution. Rajeswari and Vanjikkodi [21] developed an inventory model for deteriorating items with shortages and power demand. Sicilia et al. [22] studied the optimal replenishing policy for an inventory system with deterioration of the items and power demand pattern.

In this paper, an inventory model for deteriorating items is developed. Demand follows a power demand pattern. Shortages are allowed and partially backlogged. Products have a life-time, in which there is no deterioration. After this period, the deterioration of the items kept in stock starts. That deterioration process follows a Weibull three parametric distribution.

The work is organized as follows. Fistly, before to develop the mathematical model, the assumptions and notation used in the rest of the paper are introduced. In the third section, the costs related to the inventory control are calculated and the inventory problem is formulated. Next, the neccesary conditions to determine the optimal policy are stablished and an algorithmic approach is proposed to obtain a solution of the inventory problem. Then, some numerical examples are presented to illustrate the optimization procedure previously described. Finally, conclusions and future research lines are proposed.

HYPOTHESIS OF THE SYSTEM

The inventory system analyzed in this work has the following properties:

1. A single item is considered in the inventory system.
2. The scheduling period or inventory cycle is a decision variable of the system and it is denoted by **T.**
3. The fluctuations of the inventory level during the period **T** are continuously repeated in subsequent periods.
4. At the beginning of the inventory cycle there are **S** units in stock. That amount is unknown and it must be determined by the inventory model.
5. Lead-time is zero or negligible.
6. The length of the inventory cycle where the stock is greater than or equal to zero is another decision variable of the system and is dented by $\mathbf{t_1}$.
7. Shortages are allowed. Let **B** be the total number of shortages during the scheduling period **T**.
8. A fraction $\boldsymbol{\rho}$ of shortages are backlogged ($0<\rho\leq1$).
9. When the number of backlogged shortages is ρB, then the inventory must be replenished. The reorder point **s** is equal to $-\rho B$.
10. The replenishment is instantaneous. There is no replenishment period.
11. The replenishment size or lot size **Q** is an unknown constant and is determined by the model.
12. There exists a time period of length $\boldsymbol{\gamma}$ in which the product has no deterioration.
13. After this period $[0,\boldsymbol{\gamma}]$, it starts a process of deterioration of the product. The distribution of time to deterioration of the items follows a three-parametric Weibull distribution and the deterioration rate $\boldsymbol{\theta(t)}$ is given by

$$\theta(t) = \begin{cases} \alpha\beta(t-\gamma)^{\beta-1}, & \text{if } t \geq \gamma \\ 0, & \text{if } t < \gamma \end{cases} \qquad (1)$$

where α ($0<\alpha<1$) is the scale parameter, β (>0) is the shape parameter and γ (>0) is the location parameter.
14. There is no replacement or repair of deteriorated units during the scheduling period T.
15. The number of deteriorated units during the inventory cycle is denoted by **U.**
16. The holding cost per unit and per unit time is denoted by **h**. It is constant and known.

17. Let **w** be the backlogging cost per unit and per unit time. It is also a fixed constant.
18. Let **π** be the lost sale cost. It is a known constant.
19. The replenishing cost or ordering cost **A** is constant and independent of the ordered amount.
20. The deterioration cost associated to each deteriorated unit is denoted by **v**. It is a known constant.
21. The average demand of the product is deterministic with a rate of **r** units per inventory cycle T. The way by which quantities are taken out of the inventory to meet customer demand depends on the time when they are withdrawn. Let **f(t)** denote the product demand up to time t ($0 \le t \le T$). This demand is assumed to be the function

$$f(t) = rT\left(\frac{t}{T}\right)^{1/n} \qquad (2)$$

where n is the index of the demand pattern, with n>0. From (2), the demand rate **D(t)** at time t ($0 \le t \le T$) is

$$D(t) = \frac{rt^{1/n-1}}{nT^{1/n-1}} \qquad (3)$$

This type of demand is known as the power demand pattern (see Naddor [23], Datta y Pal [19], Lee and Wu [20], Sicilia et al. ([22], [24], [25]) and San José et al. [17]). This demand pattern describes different ways by which demanded quantities are taken out of inventory.

THE INVENTORY PROBLEM

In this section, an inventory model for a single item over an infinite horizon under deterministic demand and deterioration is developed. Let I(t) denote the net inventory level at time t, with $0 \le t \le T$. At the beginning of the inventory cycle (t=0) the replenishment of products arises the inventory level up to the maximum level S. Next, the inventory fluctuation of the system is as follows. The inventory level declines during the period $(0,\gamma]$ due to demand only. In this time period, there is no deterioration of items. Then, two scenarios can occur:

a) If the inventory level at time $t = \gamma$ is negative or equal to zero, then the life-time of the articles is long and there is no deterioration of items in the inventory system. Hence, the inventory system is reduced to the system for non-deteriorating items with a power demand pattern and partial backlogging. This system was analyzed by San José et al. [17], who proposed a general approach to obtain the optimal inventory policy (t_1^0,T^0) and the minimum cost $C^0 = C_1(t_1^0,T^0)$ per unit time, being $C_1(t_1,T)$ the total inventory cost proposed by these authors.

b) Otherwise, if the inventory level at time $t = \gamma$ is positive, then a deteriorarion process of the products takes place during the interval $(\gamma,t_1]$. In this period, the inventory level decreases due to demand and deterioration, falling to zero at time t_1. Next, shortages occur during the time period $(t_1,T]$ and a fraction ρ of shortages are backlogged. Finally, at time T, the inventory is replenished with a sufficient quantity in order to meet the backorders and leave stock to satisfy customers' demand in the next inventory cycle.

Under this second scenario, the differential equations that describe the evolution of the inventory level I(t) during the scheduling period T are

$$\frac{dI(t)}{dt} = -\frac{rt^{1/n-1}}{nT^{1/n-1}}, \qquad 0 \le t \le \gamma.$$
$$\frac{dI(t)}{dt} + \theta(t)I(t) = -\frac{rt^{1/n-1}}{nT^{1/n-1}}, \qquad \gamma \le t \le t_1$$
$$\frac{dI(t)}{dt} = -\frac{\rho rt^{1/n-1}}{nT^{1/n-1}}, \qquad t_1 \le t \le T. \qquad (4)$$

The border conditions are the following: $I(0) = S$, $I(t_1) = 0$ and $I(T) = -\rho B$. Solving the above equations, we obtain

$$I(t) = S - \frac{r}{T^{1/n-1}}t^{1/n}, \qquad 0 \le t \le \gamma$$
$$I(t) = e^{-\alpha(t-\gamma)^{\beta}}[S - \frac{r}{T^{1/n-1}}\gamma^{1/n}]$$
$$- e^{-\alpha(t-\gamma)^{\beta}}\frac{r}{nT^{1/n-1}}\int_{\gamma}^{t} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz], \qquad \gamma \le t \le t_1$$
$$I(t) = \frac{\rho r}{T^{1/n-1}}(t_1^{1/n} - t^{1/n}), \qquad t_1 \le t \le T \qquad (5)$$

As $I(t_1)=0$, this leads to

$$S = \frac{r}{T^{1/n-1}}\gamma^{1/n} + \frac{r}{nT^{1/n-1}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz \qquad (6)$$

Thus, substituting S in equation (5), we have

$$I(t) = \frac{r}{T^{1/n-1}}(\gamma^{1/n} - t^{1/n}) + \frac{r}{nT^{1/n-1}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz, \; 0 \le t \le \gamma$$
$$I(t) = \frac{r}{nT^{1/n-1}} e^{-\alpha(t-\gamma)^{\beta}}\int_{t}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz, \qquad \gamma \le t \le t_1$$
$$I(t) = \frac{\rho r}{T^{1/n-1}}(t_1^{1/n} - t^{1/n}), \qquad t_1 \le t \le T \qquad (7)$$

The maximum amount of backorders is given by

$$\rho B = -I(T) = \frac{-\rho r}{T^{1/n-1}}(t_1^{\ 1/n} - T^{1/n}) = \rho r T\left[1-\left(\frac{t_1}{T}\right)^{1/n}\right] \qquad (8)$$

The replenishing quantity or lot size Q is equal to S+ρB. Thus, from (6) and (8), the lot size is given by

$$Q = \frac{r}{nT^{1/n-1}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz + \frac{r}{T^{1/n-1}}\gamma^{1/n} + $$
$$+\frac{\rho r}{T^{1/n-1}}(T^{1/n} - t_1^{\ 1/n}) \qquad (9)$$

The amount of deteriorated items U is determined by the difference between the inventory level S at the beginning of the inventory cycle and the demanded quantity during the period $[0,t_1]$. Hence, this amount is

$$U = S - \int_0^{t_1}\frac{r}{nT^{1/n-1}}t^{1/n-1}dt = S - \frac{r}{T^{1/n-1}}t_1^{\ 1/n} =$$
$$= \frac{r}{nT^{1/n-1}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz + \frac{r}{T^{1/n-1}}(\gamma^{1/n} - t_1^{\ 1/n}) \qquad (10)$$

In this scenary, the total cost during the inventory cycle is the sum of the holding, the ordering, the deteriorating, the backlogging and the lost sale costs. In the next paragraphs, every cost is formulated as a function of the decision variables t_1 and T.

The holding cost per unit time is given by

$$\frac{h}{T}\int_0^{t_1} I(t)dt = \frac{h}{T}\int_0^{\gamma} I(t)dt + \frac{h}{T}\int_{\gamma}^{t_1} I(t)dt =$$
$$= \frac{hr\gamma^{1/n+1}}{(n+1)T^{1/n}} + \frac{hr\gamma}{nT^{1/n}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz +$$
$$+\frac{hr}{nT^{1/n}}\int_{\gamma}^{t_1} e^{-\alpha(t-\gamma)^{\beta}}\left(\int_{t}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz\right)dt \qquad (11)$$

The ordering cost per unit time is A/T.

The deteriorated units cost per unit time is

$$\frac{vU}{T} = \frac{vr}{nT^{1/n}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz + \frac{vr}{T^{1/n}}(\gamma^{1/n} - t_1^{\ 1/n}) \qquad (12)$$

The backlogging cost per unit time is

$$\frac{w}{T}\int_{t_1}^{T}(-I(t))dt = \frac{w}{T}\int_{t_1}^{T}\frac{-\rho r}{T^{1/n-1}}(t_1^{\ 1/n} - t^{1/n})\,dt =$$
$$= \frac{\rho wr}{T^{1/n}}\int_{t_1}^{T}(t^{1/n} - t_1^{\ 1/n})\,dt =$$
$$= \frac{\rho wrn}{n+1}T + \frac{\rho wr}{(n+1)T^{1/n}}t_1^{\ 1/n+1} - \frac{\rho wr}{T^{1/n-1}}t_1^{\ 1/n} \qquad (13)$$

From (8), the total shortage during the period $[t_1,T]$ is given by

$$B = \frac{-I(T)}{\rho} = \frac{r(T^{1/n} - t_1^{\ 1/n})}{T^{1/n-1}} \qquad (14)$$

The number of lost sales is (1-ρ)B. Thus, the lost sale cost is given by

$$\frac{\pi(1-\rho)B}{T} = \frac{\pi(1-\rho)r(T^{1/n} - t_1^{\ 1/n})}{T^{1/n}} \qquad (15)$$

Therefore, the total cost per unit time is given by the sum of the above costs, that is

$$C_2(t_1,T) = \frac{hr\gamma^{1/n+1}}{(n+1)T^{1/n}} + (h\gamma+v)\frac{r}{nT^{1/n}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz +$$
$$+\frac{hr}{nT^{1/n}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}\left(\int_{\gamma}^{z} e^{-\alpha(t-\gamma)^{\beta}}dt\right)dz + \frac{A}{T} +$$
$$+\frac{vr}{T^{1/n}}\gamma^{1/n} - [v+\pi(1-\rho)]\frac{r}{T^{1/n}}t_1^{\ 1/n} + \pi(1-\rho)r +$$
$$+\frac{\rho wrn}{n+1}T + \frac{\rho wr}{(n+1)T^{1/n}}t_1^{\ 1/n+1} - \frac{\rho wr}{T^{1/n-1}}t_1^{\ 1/n} \qquad (16)$$

The inventory problem consists of minimizing the cost function (16) subject to the constraints $0 \le t_1 \le T$ and $T > 0$.

NECCESARY CONDITIONS TO DETERMINE THE OPTIMAL POLICY

To find the neccesary conditions of optimality of the inventory problem we have to calculate the partial derivatives of the function $C_2(t_1,T)$ with respect to t_1 and T. Thus, we have

$$\frac{\partial C_2}{\partial t_1} = (h\gamma+v)\frac{r}{nT^{1/n}}e^{\alpha(t_1-\gamma)^{\beta}}t_1^{\ 1/n-1} +$$
$$+\frac{hr}{nT^{1/n}}t_1^{\ 1/n-1}e^{\alpha(t_1-\gamma)^{\beta}}\int_{\gamma}^{t_1} e^{-\alpha(t-\gamma)^{\beta}}dt - \frac{\rho wrt_1^{\ 1/n-1}}{nT^{1/n-1}} +$$
$$+\frac{\rho wrt_1^{\ 1/n}}{nT^{1/n}} - \frac{[\pi(1-\rho)+v]r}{nT^{1/n}}t_1^{\ 1/n-1} \qquad (17)$$

$$\frac{\partial C_2}{\partial T} = -\frac{r}{nT^{1/n+1}}\left[\frac{h\gamma^{1/n+1}}{n+1} + \frac{(h\gamma+v)}{n}\int_{\tau}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}dz\right] +$$
$$-\frac{hr}{n^2T^{1/n+1}}\int_{\gamma}^{t_1} e^{\alpha(z-\gamma)^{\beta}} z^{1/n-1}\left(\int_{\gamma}^{z} e^{-\alpha(t-\gamma)^{\beta}}dt\right)dz + -\frac{A}{T^2} +$$
$$+\left[-v\gamma^{1/n} - \frac{\rho wt_1^{\ 1/n+1}}{n+1} + (1-n)\rho wTt_1^{\ 1/n}\right]\frac{r}{nT^{1/n+1}}$$
$$+\frac{\rho wrn}{n+1} + \frac{r}{nT^{1/n+1}}[\pi(1-\rho)+v]t_1^{\ 1/n} \qquad (18)$$

Setting the partial derivative (17) equal to zero, we obtain the first condition

$$T = t_1 + \frac{h}{\rho w} e^{\alpha(t_1-\gamma)^\beta} \int_\gamma^{t_1} e^{-\alpha(t-\gamma)^\beta} dt + \frac{(h\gamma + v)}{\rho w} e^{\alpha(t_1-\gamma)^\beta} - \frac{[\pi(1-\rho)+v]}{\rho w} \tag{19}$$

Also, setting the partial derivative (18) equal to zero, we have the second condition

$$\frac{hr\gamma^{1/n+1}}{n+1} + \frac{r(h\gamma+v)}{n} \int_\gamma^{t_1} e^{\alpha(z-\gamma)^\beta} z^{1/n-1} dz + \frac{hr}{n} \int_\gamma^{t_1} e^{\alpha(z-\gamma)^\beta} z^{1/n-1} (\int_\gamma^{z} e^{-\alpha(t-\gamma)^\beta} dt) dz + r v \gamma^{1/n} + \frac{r\rho w t_1^{1/n+1}}{n+1} - [r\pi(1-\rho) + rv] t_1^{1/n} + AnT^{1/n-1} - \frac{\rho w r n^2}{n+1} T^{1/n+1} - (1-n) r \rho w t_1^{1/n} T = 0 \tag{20}$$

Substituting the value of T given by (19) in the equation (20) leads to the following non-linear equation with an unique variable t_1.

$$\frac{hr\gamma^{1/n+1}}{n+1} + \frac{r(h\gamma+v)}{n} \int_\gamma^{t_1} e^{\alpha(z-\gamma)^\beta} z^{1/n-1} dz + r v \gamma^{1/n} + \frac{r\rho w t_1^{1/n+1}}{n+1} + \frac{hr}{n} \int_\gamma^{t_1} e^{\alpha(z-\gamma)^\beta} z^{1/n-1} (\int_\gamma^{z} e^{-\alpha(t-\gamma)^\beta} dt) dz - [r\pi(1-\rho) + rv] t_1^{1/n} + An[G(t_1)]^{1/n-1} - \frac{\rho w r n^2}{n+1} [G(t_1)]^{1/n+1} - (1-n) r \rho w t_1^{1/n} [G(t_1)] = 0 \tag{21}$$

being

$$G(t_1) = t_1 + \frac{h}{\rho w} e^{\alpha(t_1-\gamma)^\beta} \int_\gamma^{t_1} e^{-\alpha(t-\gamma)^\beta} dt + \frac{(h\gamma + v)}{\rho w} e^{\alpha(t_1-\gamma)^\beta} - \frac{[\pi(1-\rho)+v]}{\rho w}$$

The equation (21) has to be solved by using an numerical procedure. For example, it can be used the bisection method. Thus, the time period t_1* can be determined and, substituting this value in the equation (19), leads to the inventory cycle T*.

The second partial derivatives of $C_2(t_1,T)$ are

$$\frac{\partial^2 C_2}{\partial t_1^{\,2}} = (h\gamma + v)\frac{r}{nT^{1/n}} e^{\alpha(t_1-\gamma)^\beta} t_1^{\,1/n-2} \left[\alpha\beta(t_1-\gamma)^{\beta-1} t_1 + \frac{(1-n)}{n}\right] + \frac{hr}{nT^{1/n}} e^{\alpha(t_1-\gamma)^\beta} t_1^{\,1/n-2} (\int_\gamma^{t_1} e^{-\alpha(t-\gamma)^\beta} dt) \left[\alpha\beta(t_1-\gamma)^{\beta-1} t_1 + \frac{(1-n)}{n}\right] + \frac{(\rho w + nh) r t_1^{\,1/n-1}}{n^2 T^{1/n}} - \frac{(1-n)\rho w r t_1^{\,1/n-2}}{n^2 T^{1/n-1}} - (\pi(1-\rho)+v)\frac{(1-n) r t_1^{\,1/n-2}}{n^2 T^{1/n}} \tag{22}$$

$$\frac{\partial^2 C_2}{\partial t_1 \partial T} = -\frac{r t_1^{\,1/n-1}}{n^2 T^{1/n+1}} \left[(h\gamma + v) e^{\alpha(t_1-\gamma)^\beta} + h e^{\alpha(t_1-\gamma)^\beta} (\int_\gamma^{t_1} e^{-\alpha(t-\gamma)^\beta} dt)\right] + - \frac{r t_1^{\,1/n-1}}{n^2 T^{1/n+1}} \left[-(v + \pi(1-\rho)) + \rho w t_1 - (1-n)\rho w T\right] \tag{23}$$

$$\frac{\partial^2 C_2}{\partial T^2} = \frac{(n+1)r}{n^2 T^{1/n+2}} \left[\frac{(h\gamma+v)}{n} \int_\gamma^{t_1} e^{\alpha(z-\gamma)^\beta} z^{1/n-1} dz\right] + \frac{hr\gamma^{1/n+1}}{n^2 T^{1/n+2}} + \frac{2A}{T^3} + \frac{h(n+1)r}{n^3 T^{1/n+2}} \int_\gamma^{t_1} e^{\alpha(z-\gamma)^\beta} z^{1/n-1} (\int_\gamma^{z} e^{-\alpha(t-\gamma)^\beta} dt) dz + \frac{vr(n+1)\gamma^{1/n}}{n^2 T^{1/n+2}} + \frac{r t_1^{\,1/n}}{n^2 T^{1/n+2}} [\rho w t_1 - (v + \pi(1-\rho))(n+1) - \rho w (1-n) T] \tag{24}$$

Let (t_1*,T*) be the solution point of equations (19) and (21). This point is a minimum of the function $C_2(t_1,T)$ if the Hessian matrix is positive definite. The second partial derivatives at point (t_1*,T*) are

$$\frac{\partial^2 C_2(t_1^*, T^*)}{\partial t_1^{\,2}} = \frac{r(t_1^*)^{1/n-1}}{n(T^*)^{1/n}} [h + \rho w] + \frac{r(t_1^*)^{1/n-1}}{n(T^*)^{1/n}} \alpha\beta(t_1^* - \gamma)^{\beta-1} [v + \pi(1-\rho) + \rho w (T^* - t_1^*)] \tag{25}$$

$$\frac{\partial^2 C_2(t_1^*, T^*)}{\partial t_1 \partial T} = -\frac{\rho w r (t_1^*)^{1/n-1}}{n(T^*)^{1/n}} \tag{26}$$

$$\frac{\partial^2 C_2(t_1^*, T^*)}{\partial T^2} = \frac{\rho w r (1-n)(t_1^*)^{1/n}}{n(T^*)^{1/n+1}} + \frac{(n-1)A}{n(T^*)^3} + \frac{\rho w r}{T^*} \tag{27}$$

From (25), leads to the second partial derivative with respect to t_1 is positive at the point (t_1*, T*). Thus, this point is the optimal policy of the system if the Hessian at point (t_1*, T*) is positive, that is

$$H(t_1^*,T^*) = \frac{\partial^2 C_2(t_1^*,T^*)}{\partial t_1^2}\frac{\partial^2 C_2(t_1^*,T^*)}{\partial T^2} - \left[\frac{\partial^2 C_2(t_1^*,T^*)}{\partial t_1 \partial T}\right]^2 > 0 \quad (28)$$

ALGORITHM

The following procedure determines the optimal policy for an inventory system with a power demand pattern, where shortages are allowed. There exists a life-time period γ without decay in the quality of items and a posterior deterioration process with Weibull distribution that takes place after the life-time period of items.

Step 1. Following the procedure proposed by San José et al. [17], to determine the optimal inventory policy (t_1^0,T^0) for the non-deterioration system. Calculate the minimum cost $C^0 = C_1(t_1^0,T^0)$ per time unit.

Step 2. If t_1^0 is greater than γ, go to Step 4. Otherwise, go to Step 3.

Step 3. The policy proposed by (t_1^0,T^0) is the optimal solution of the non-deterioration inventory problem. The minimum cost is given by $C^0 = C_1(t_1^0,T^0)$. Stop.

Step 4. Using some numerical method, to determine the set Ω of solutions t_1 of the equation (21) such that $t_1 > \gamma$. Choose an element t_1 of the set Ω.

Step 5. From (19), to calculate the value of T associated to t_1. From (28), get the value of the Hessian at point (t_1,T).

Step 6. - If $H(t_1,T)>0$, then include the pair (t_1,T) into the set P of candidate inventory policies. From (16), to calculate the cost $C_2(t_1,T)$. Go to step 7.
- Otherwise, go to step 7.

Step 7. Set $\Omega = \Omega - \{t_1\}$.
- If Card(Ω) = 0, then go to step 8.
- Otherwise, to choose a new positive solution t_1 of the set Ω. Go to step 5.

Step 8. Determine the pair (t_1^*,T^*) such that its cost C* is the lowest cost of the policies included in P. That pair is the optimal policy. Stop.

The algorithmic procedure allows also determining the optimal inventory policy for a system with a deterioration process that stars at the beginning of the inventory cycle. For that, set $\gamma = 0$ in all the formulas and equations where γ appears, and starts the application of the algorithm from step 4.

NUMERICAL EXAMPLES

Example 1. Consider an inventory system for an ecological refreshing ice cream that has neither preservatives nor additives. The life-time of the product is γ = 3 days. Assume that the system satisfies the properties described in Section 2, and suppose the following parameters: average demand rate r = 100 liters per week, power demand index n =2, scale parameter α = 0.1, shape parameter β = 1, ordering cost A = 40 $, unit holding cost h = 3 $ per liter and day, unit deterioration cost v = 20 $ per liter and day, unit backlogging cost w = 10 $ per liter and day, lost sale cost π = 20 $ per liter and day, and backlogging rate ρ =1. Applying the algorithm, leads to

Step 1. $t_1^0 = 0.537381$.
Step 2. $t_1^0 > 3/7 = \gamma = 0.428571$.
Step 4. $\Omega = \{0.497451\}$.
Step 5. T = 0.661470, $H(t_1,T) = 2030.50$
Step 6. $P = \{(t_1,T)\}$. $C_2(t_1,T) = 115.213$
Step 8. The optimal policy is given by $t_1^* = 0.497451$ weeks, T* = 0.661470 weeks, and the minimum cost is C* = 115.213 $ per week.

Example 2. Suppose the same parameters that the above example, but changing the power demand index to n = 0.5. Following the algorithm, we have

Step 1. $t_1^0 = 0.417029$, $T^0 = 0.542137$, $C^0 = 147.564$.
Step 2. $t_1^0 \leq 3/7 = \gamma = 0.428571$.
Step 3. The optimal policy is the pair (t_1^0,T^0) and the minimum cost per unit time is C^0 = 147.564 $ per week.

Example 3. Assume the same parameters that the Example 1, but changing the life-time to γ = 2 days. Applying the algorithm, leads to

Step 1. $t_1^0 = 0.537381$.
Step 2. $t_1^0 > 2/7 = \gamma$.
Step 4. $\Omega = \{0.449512\}$.
Step 5. T = 0.619216, $H(t_1,T) = 55559.3$
Step 6. $P = \{(t_1,T)\}$. $C_2(t_1,T). = 118.665$
Step 8. The optimal policy is $t_1^* = 0.449512$ weeks, T* = 0.619216 weeks, and the minimum cost is C* = 118.665 $ per week.

CONCLUSIONS

In this work, an economic ordering quantity (EOQ) model for an inventory system with a power demand pattern and partial backlogging have been analyzed. In this system it is considered that there is a time period in which the items kept in stock are perfectly maintained and do not suffer any type of loss or decrease in the quality of them. After that time, there is a process of deterioration in the stored items that causes a percentage of these products to be spoiled and cannot be sold.

Two scenarios are introduced. In the first one, it is supposed that the life-time of items is quite long such that the articles have not deterioration when they are in the inventory. In this case, the inventory system is described by an inventory model without deterioration, with a power demand pattern and partial backlogging. This system was analyzed by San José et al. [17]. They proposed an efficient procedure to determine the optimal inventory policy.

The other scenario is detailed in this paper. In this case, it is assumed that the life-time of the articles is less than the period of stay of the same in the inventory. Thus, the fluctuations of the inventory level are described as follows. At the beginning, the inventory is replenished to meet the future demand of customers. In a first time period, the inventory level decays due to demand only. After the life-period of items, a deterioration process of products takes place. Thus, in a second period, the inventory level declines due to demand and deterioration, falling to zero level at time t_1. Next, as there is no stock in the inventory, shortages appear in the system and they are partially backlogged at the end of the inventory cycle with the entry of the next replenishing.

A mathematical model describing the inventory system is introduced and then an optimal decision procedure is derived from the model. The different types of costs of the inventory system are presented and the objective function to optimize is established. The necessary conditions to obtain the optimal inventory policy are developed. Then, an algorithmic approach is proposed to find what values of the decision variables will give the minimum total inventory cost per unit time. Also, some numerical examples are considered to illustrate the theoretical results.

Finally, we can cite the following future research directions: (i) to study the system with a power demand pattern, deterioration, shortages and uniform replenishing rate; (ii) to analyze the inventory system for deterioration items, assuming power demand, shortages and quantity discounts; (iii) to develop the system with power demand, deterioration, shortages and time-dependent non-constant holding cost; and (iv) to study the inventory system for deteriorating items, considering a power demand pattern, shortages and a time-dependent backlogging rate.

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