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Analytical Formula of European-Style Power Call Options in an MFBM with Jumps Model

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Abstract: Studies have shown that stock price process exhibits long-range dependence. To address this, many have introduced the mixed-fractional Brownian motion (MFBM) model to the stock price process. Under risk-neutral measure, this study provides an analytical formula for the price of European-style power call options in an MFBM environment with the inclusion of the jumps process. Modeling the stock price with MFBM and jumps process enables the capturing of long memory trend as well as discontinuity in the stock price process.

Keywords: European-style options, Power Calls, Mixed-Fractional Brownian Motion, Jumps Process

1. Introduction

Options are financial contracts that can be classified as American-style options which can be exercised at any time before their maturity dates, or European-style options which can be exercised at their maturity dates. American-style and European-style options can be further categorized as being either call options which give the holders the rights but not the obligations to buy underlying assets for a certain strike price at their maturities or put options which give the holders the rights but not the obligations to sell underlying assets for a certain strike price at their maturities. Since they were introduced, they have been extensively traded.

The most common options are ordinary options, also known as vanilla options. These options have been modified and improved to produce various types of exotic options to fulfill investors' needs, such as power options. At maturity, for a predetermined strike, the holder of a power call option has the right, but not the obligation, to buy the underlying assets collectively; whereas the holder of a power put option has the right, but not the obligation, to sell the underlying assets [14]. The payoff of a power call option is described by raising the underlying asset to some fixed integer b > 1 as such:

$$PC_{payoff} = \max(S_T^b - K, 0) \tag{1}$$

Studies on power options are significant theoretically and practically. The flexibility and higher leverage of power options attract investors since a minor adjustment in the underlying asset price results in a significant change in the prices. Hence, power options have been traded in the markets, such as derivatives on commodities and foreign exchanges [2-5]. Moreover, Bankers Trust in Germany issues power options of order 2 [6], and polynomial options are issued on the Nikkei index [2].

References [7] and [8] have studied the pricing of power options within the Black-Scholes [9] framework. Nevertheless, studies by [10-12] show that the distribution of the logarithmic returns exhibits features like fat tails, volatility smile, and long-range dependence. This motivates other studies to extend the Black-Scholes [9] model to pricing power options problems, for instance, by incorporating stochastic volatility, such as the work of [13],

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and including the jump-diffusion process, such as [14] and [15]. Moreover, [16] models the power option valuation problem for an uncertain financial market, while recently, [17] provides a model-free approach to power option pricing.

The distribution of the logarithmic returns has been described with fractional Brownian motion [18-21]; however, it allows arbitrage opportunities [22,23]. The work of [24] and [25] introduced the mixed-fractional Brownian motion process, where [26] shows that by choosing $H \in \left(\frac{3}{4}, 1\right)$, this reduces to the standard Brownian motion which avoids arbitrage.

According to [1], the mixed-fractional Brownian motion model can capture the local variability of the process model; hence gives a more realistic view of financial data. It has also contributed to the option pricing problem [27-30], warrants pricing [31,32] and convertible bonds [33]. Some studies have included jumps, such as in [32-35].

This study aims to model and derive an analytical pricing formula for European-style power call options within the mixed-fractional Brownian motion environment with jumps process. The organization of this paper is as follows. Section 2 briefly describes the stock price process in the mixed-fractional Brownian motion environment with jumps model. Section 3 derives the analytical solution for European-style power call options, and Section 4 concludes the paper.

2. Mixed-Fractional Brownian Motion with Jumps Model

This section briefly describes the mixed-fractional Brownian motion (MFBM hereon) with jumps model where the jumps process follows the dynamics in [36].

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_{t,t\geq 0}\}, \mathbb{P}\right)$ be a filtered probability space where $\mathbb{P}$ is the physical probability measure. Then an MFBM $B_t^H(\alpha, \beta)$ is defined by:

$$B_t^H(\alpha, \beta) = \alpha W_t + \beta W_T^H \tag{2}$$

where W_t is a Brownian motion, W_t^H is a fractional Brownian motion with Hurst parameter $H \in \left(\frac{3}{4}, 1\right)$, $\{\mathcal{F}_{t,t\geq 0}\}$ is the $\mathbb{P}$-augmentation of the filtration generated by (W_τ, W_τ^H) for $\tau \leq t$\$ and α and β are some real non-zero constants. The properties of MFBM can be referred to in [27,28].

Assuming a frictionless market, under a risk-neutral framework, the stock price process under MFBM and jumps is as follows:

$$dS_t = (r - \lambda k) S_t dt + \sigma S_t dW_t + \varepsilon S_t dW_t^H + (y_t - 1) S_t dN_t \tag{3}$$

where r is the risk-free rate, σ and ε are constant volatility of the logarithmic returns when jumps do not occur, N_t represents a Poisson process with intensity λ, y_t is the size of absolute price jump, and $(y_t - 1)$ is the size of the relative price jump [36]. The source of randomness, W_t, W_t^H, N_t, and y_t are assumed to be independent.

Reference [36] describes the logarithmic asset price jump sizes are normally distributed $\ln y_t \sim \mathcal{N}(\mu_J, \delta^2)$, where μ_J and δ are i.i.d. logarithmic return jump size and i.i.d. volatility of the logarithmic return jump, respectively. Therefore:

$$(y_t - 1) \sim Lognormal\left(e^{\mu_J + \frac{1}{2}\delta^2} - 1, e^{2\mu_J + \delta^2}\left[e^{\delta^2} - 1\right]\right).$$

On that account, by using Itô formula, the solution to SDE (3) is obtained as follows:

$$S_T = S_t e^{\left[\left(r - \lambda k - \frac{1}{2}\sigma^2\right)(T-t) - \frac{1}{2}\varepsilon^2\left(T^{2H} - t^{2H}\right) + \sigma(W_T - W_t) + \varepsilon\left(W_T^H - W_t^H\right) + \sum_{i=1}^{N_t} \ln y_i\right]}, \tag{4}$$

where $k \equiv \mathbb{E}(y_1 - 1) = e^{\mu_J + \frac{1}{2}\delta^2} - 1$, is the expectation of the changes in the asset price due to a shock (or jump) in the market. Moreover, Equation (4) can be represented as the following:

$$\ln\left(\frac{S_T}{S_t}\right) = \left(r - \lambda k - \frac{1}{2}\sigma^2\right)(T - t) - \frac{1}{2}\varepsilon^2(T^{2H} - t^{2H}) + \sigma(W_T - W_t) + \varepsilon(W_T^H - W_t^H) + \sum_{i=1}^{N_t} \ln y_i. \tag{5}$$

We can now derive the analytical formula for European-style power call options.

3. Analytical Formula for European-Style Power Call Options

This section derives the analytical formula for European-style power call options for the payoff function $H(S_T^b)$ under mixed-fractional Brownian motion (MFBM hereon) with jumps model.

The price of the European-style power call option is expressed as the discounted risk-neutral conditional expectation of its payoff function at a risk-free rate r as follows:

$$PC(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[H(S_T^b) \middle| \mathcal{F}_t\right], \tag{6}$$

which can be written as such:

$$PC(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[H\left(S_T^b e^{\ln\left(\frac{S_T^b}{S_t^b}\right)}\right) \middle| \mathcal{F}_t\right], \tag{7}$$

or similarly:

$$PC(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[H\left(S_T^b e^{b \ln\left(\frac{S_T}{S_t}\right)}\right) \middle| \mathcal{F}_t\right]. \tag{8}$$

By letting $Y_k = \ln y_k$ and substituting (5) into (8) yields:

$$PC(t, S_t) = e^{-r(T-t)} \times \mathbb{E}^{\mathbb{Q}}\left[H\left(S_T^b e^{b\left[\left(r - \lambda k - \frac{1}{2}\sigma^2\right)(T-t) - \frac{1}{2}\varepsilon^2\left(T^{2H} - t^{2H}\right) + \sigma(W_T - W_t) + \varepsilon\left(W_T^H - W_t^H\right) + \sum_{k=1}^{N_{T-t}} Y_k\right]}\right)\right]$$

Moreover, for i number of jumps, the given distribution of a compound Poisson process is $\sum_{k=1}^{N_{T-t}} Y_k \sim \mathcal{N}(i\mu_J, i\delta^2)$, where $i \equiv N_{T-t} = 0,1,2,\ldots$ Thus, by using the law of iterated expectations and conditioning on i, we have the following:

$$PC(t,S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[H(S_T^b e^{b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})\sigma(W_T-W_t)+\varepsilon(W_T^H-W_t^H)+\sum_{k=1}^{N_{T-t}}Y_k\right]}|N_{T-t} = i)]]$$

$$= e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)[\mathbb{E}^{\mathbb{Q}}[H(S_T^b e^{b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})\right]}$$

$$\times e^{b\left[\sigma(W_T-W_t)+\varepsilon(W_T^H-W_t^H)+\sum_{k=1}^{N_{T-t}}Y_k\right]}|N_{T-t} = i)]]$$

where:

$$b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})+\sigma(W_T-W_t)+\varepsilon(W_T^H-W_t^H)+\sum_{k=1}^{i}Y_k\right]$$

$$\sim \mathcal{N}\left(b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})+i\mu\right]; b^2[\sigma^2(T-t)+\varepsilon^2(T^{2H}-t^{2H})+i\delta^2]\right).$$

Let:

$$m = b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})+i\mu\right],$$

$$s = b\sqrt{\sigma^2(T-t)+\varepsilon^2(T^{2H}-t^{2H})+i\delta^2}.$$

This implies that $\ln\ln\left(\frac{S_T^b}{S_t^b}\right) \sim \mathcal{N}(m, s^2) = m + sz.$
Given the payoff function (1), we obtain:

$$PC(t,S_t) = e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\mathbb{E}^{\mathbb{Q}}\left[\max\left(S_t^b e^{\ln\left(\frac{S_T^b}{S_t^b}\right)}-K,0\right)\right]$$

$$= e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\left[\mathbb{E}^{\mathbb{Q}}\left(S_t^b e^{\ln\left(\frac{S_T^b}{S_t^b}\right)}\mathbf{I}_{\{S_T^b>K\}}\right)-K\mathbb{E}^{\mathbb{Q}}\left(\mathbf{I}_{\{S_T^b>K\}}\right)\right] \quad (9)$$

$$= e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)[\mathbb{E}^{\mathbb{Q}}(S_t^b e^{m+sz}\mathbf{I}_{\{z>d_1\}})-K\mathbb{E}^{\mathbb{Q}}(\mathbf{I}_{\{z>d_1\}})]$$

where $d_1 = \frac{\ln\left(\frac{K}{S_t^b}\right)-m}{s}$. Following that, we have:

$$PC(t,S_t) = e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\left[S_t^b e^m\int_{d_1}^{\infty}e^{sz}\left(\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\right)dz-K\int_{d_1}^{\infty}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz\right]$$

$$= e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\left[S_t^b e^{m+\frac{1}{2}s^2}\int_{d_1}^{\infty}\left(\frac{e^{-\frac{1}{2}(z-s)^2}}{\sqrt{2\pi}}\right)dz-K\int_{d_1}^{\infty}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz\right] \quad (10)$$

Suppose we let $v = z - s$, then we have:

$$PC(t,S_t) = e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\left[S_t^b e^{m+\frac{1}{2}s^2}\int_{d_1-s}^{\infty}\left(\frac{e^{-\frac{1}{2}v^2}}{\sqrt{2\pi}}\right)dv-K\int_{d_1}^{\infty}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz\right]$$

$$= e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\left[S_t^b e^{m+\frac{1}{2}s^2}[1-N(d_1-s)]-K[1-N(d_1)]\right] \quad (11)$$

$$= e^{-r(T-t)}\left(\sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\right)\left[S_t^b e^{m+\frac{1}{2}s^2}[N(-d_1+s)]-K[N(-d_1)]\right]$$

This leads to our desired result given in the following theorem.

Theorem 1. The price of European-style power call options with time to maturity T and strike K, under MFBM with jumps model (3) is given by:

$$PC(t,S_t) = \sum_{i=0}^{\infty}\frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!}\left[S_t^b e^{(b-1)\left(r+\frac{1}{2}b\sigma^2\right)(T-t)}N(d_1^J)-KN(d_2^J)\right] \quad (12)$$

where:

$$d_1^J = b\sqrt{\sigma^2(T-t)+\varepsilon^2(T^{2H}-t^{2H})+i\delta^2} + \left[\frac{\ln\left(\frac{S_t^b}{K}\right)+b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})+i\mu\right]}{b\sqrt{\sigma^2(T-t)+\varepsilon^2(T^{2H}-t^{2H})+i\delta^2}}\right],$$

$$d_2^J = \frac{\ln\left(\frac{S_t^b}{K}\right)+b\left[\left(r-\lambda k-\frac{1}{2}\sigma^2\right)(T-t)-\frac{1}{2}\varepsilon^2(T^{2H}-t^{2H})+i\mu\right]}{b\sqrt{\sigma^2(T-t)+\varepsilon^2(T^{2H}-t^{2H})+i\delta^2}},$$

$$\hat{\lambda} = \lambda\left(e^{\mu_J+\frac{1}{2}\delta^2}\right),$$

$$k = e^{e^{\mu_J+\frac{1}{2}\delta^2}} - 1.$$

We observe that as $b \to 1$, the pricing solution (12) reduces to the European-style vanilla call options pricing formula in an MFBM with jumps model.

4. Conclusion

This study aims to derive an analytical pricing formula for European-style power call options under a mixed-fractional Brownian motion environment with jumps. This model captures long-memory phenomenon and discontinuous behavior in the logarithmic returns. Further studies may include incorporating stochastic interest rates or stochastic volatility to the dynamics of the asset price process.

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